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NATURAL CONVECTION NEAR AND ABOVE THERMAL LEADING EDGES ON VERTICAL WALLS

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Abstract—The natural convection from the neighborhood of several types of thermal leading edges on vertical walls is investigated. The influence of leading edge conditions for uniform and special variable wall temperature cases is determined. It is shown for the constant temperature case that leading edge conditions effect, at suitable distances above the leading edge, only the relative vertical position of the boundary layer and not its thermal or velocity form. Universal curves which may be used with relatively arbitrary initial boundary conditions are presented for the uniform wall temperature case at Prandtl number 0·7. It is indicated how these curves may be used in a piece-wise manner to investigate relatively arbitrary wall temperature cases.

NOMENCLATURE

- c_p, specific heat at constant pressure;
- k, thermal conductivity;
- x_0, non-dimensionalizing length;
- $Nu_x = - x(\partial T/\partial y)/(T_w - T_\infty)$, local Nusselt number;
- $Gr_x = g\beta x^3(T_w - T_\infty)/\nu$, local Grashof number;
- β, coefficient of thermal expansion;
- T, absolute temperature;
- x, y, vertical, horizontal co-ordinates with origin at bottom surface point;
- μ, ν, absolute and kinematic viscosity respectively;
- g, acceleration due to gravity;
- ρ, fluid density;
- Pr, Prandtl number;
- τ, temperature function;
- δ, boundary layer thickness;
- p, temperature function coefficient;
- u, speed function;
- U, speed.

Subscripts and superscripts

- w, wall conditions;
- ∞, ambient conditions far from wall;
- $-$, non-dimensionalized quantity.

THE heat transfer in natural convection from heated or cooled vertical plate or wall sections has been treated by methods which are generally variations on the method introduced by Pohlhausen [1] and that credited by Goldstein [2] to an unpublished paper by Squire. Both of these methods do not take cognizance of the boundary conditions at or near the thermal leading edge in the development of the solutions, and the results are generally unsuitable in this region and questionable above it. For example, one finds in the case of the constant temperature vertical plate infinite heat transfer at this leading edge and a convergence of the isotherms to this point. In fact, it will be shown that the absence of real initial boundary conditions makes the previous solution for this case rather fortuitous. This paper treats both the constant and variable temperature plates and includes appropriate initial boundary conditions.

General equations for vertical plate

The integrated momentum–energy boundary layer equations may be used. They are respectively,

$$- g \int_0^\delta (\rho_\infty - \rho)\, dy + \frac{d}{dx} \int_0^\delta \rho U^2\, dy + \mu \left(\frac{\partial U}{\partial y}\right)_{y=0} = 0, \quad (1)$$

2Y

$$c_p\, g\, \frac{d}{dx}\int_0^{\delta} \rho U(T - T_\infty)\, dy + k(\partial T/\partial y)_{y=0} = 0. \quad (2)$$

It will be seen later that in one case of variable wall temperatures the energy equation should also have included a vertical conduction term for the bottom region. It is left out for the moment because it does not appear to be significant in the resulting affect above the bottom region.

An extension of the Squire method will be used and we will seek similarity solutions of the form

$$U = u(x)(\eta^2 - \eta^3), \quad \eta = 1 - y/\delta \quad (3)$$

$$T/T_\infty = 1 + \tau(x)\eta^2. \quad (4)$$

The boundary conditions at the plate surface, $U = 0$; $T/T_\infty = \tau(x) + 1$ and at the ambient isotherm $U = 0$; $T/T_\infty = 1$ are evidently satisfied. The initial boundary conditions are yet to be imposed. The solution forms (3) and (4) reduce the equations (1) and (2) to the ordinary differential equations:

$$\tau' u\delta^2 + \tau(u'\delta^2 + u\delta\delta' - C_2) = 0 \quad (5)$$

$$2uu'\delta^2 + u^2\delta\delta' - C_3\tau\delta^2 + C_4 u = 0 \quad (6)$$

in which

$$C_2 = 60k/\rho c_p g = 60\nu/Pr; \quad C_3 = 35g; \quad C_4 = 105\nu \quad (7)$$

and a term $\tau/252$ has been dropped as negligible in comparison to a term 1/105.

The constant temperature case

Several geometries must be examined in the study of the so-called constant temperature case. One that has been used in the experimental treatment of the problem is the vertically suspended plate (Fig. 1a). The equations which have been used and also those in this paper are not applicable to the leading edge geometry below the vertical flat surfaces because for one thing of the relative direction of the buoyancy forces and vertical velocity components to the surface. A complete solution of the problem would require a multiple matching-solution system analogous to the two-solution system used for the flat plate in a uniform stream [3]. The equations above are applicable only at and above the indicated y-axis and the initial boundary values $\delta = \delta_0$, $u = u_0$ to be used with them must be determined independently. These initial values would, for example, be part of the solution matching process. For the present purpose, the values δ_0, u_0 are treated as arbitrary initial values. The solution of (5) and (6) with the required arbitrary initial boundary condition will be given below.

A second geometry (see Fig. 1b) that may be considered is that of a vertical wall which has a constant temperature T_w down to a point $x = 0$ and the constant wall temperature T_∞ (the ambient temperature) below this point. Even if the wall materials permitted a good approximation to such a temperature jump, there would none the less be a downward conduction of heat in the fluid near the wall and a resultant region of variable temperature below the point $x = 0$ which would once again require a separate

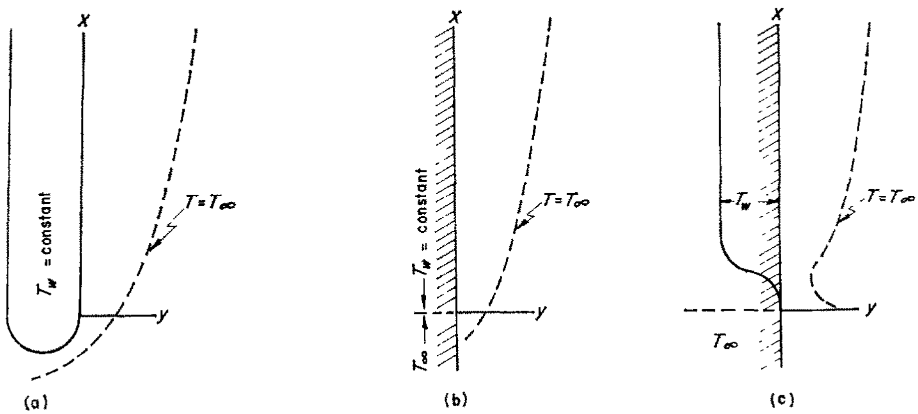

FIG. 1 (a) (b) (c). Constant temperature cases considered.

investigation. Again equations (5) and (6) are solvable for $x \geqslant 0$ when initial boundary conditions $\delta(0) = \delta_0$, $u(0) = u_0$ are given.

A third geometry which may be considered is that shown in Fig. 1(c). In that case, the wall temperature down to a point on the wall $x = x_0$ is a constant $T_w = T_0$. From $x = x_0$ to $x = 0$. T_w is assumed to continuously decay to the ambient temperature T_∞ in such a way that $\mathrm{d}T_w/\mathrm{d}x = 0$ at $x = 0$ so that there will be no downward vertical conduction of heat at $x = 0$. An initial boundary condition is definitely $U = 0$ and $x = 0$ is a part of the ambient isotherm. The bottom region does not need independent treatment from the standpoint of initial geometry as in the first two cases and a complete solution is available from equations (5) and (6). Even in this case, however, further examination of the results in the leading edge region will be needed to determine the effect of terms omitted in the boundary layer equations used.

Solutions for the first two cases

It happens that a particularly simple solution is available for these cases if the initial boundary conditions are related by the equation $(u_0/a_0) = (\delta_0/b_0)^2$ in which

$$\left.\begin{aligned} a_0^2 &= 4C_3\tau/(5 + 3C_4/C_2) \\ &= 560g\tau/(20 + 21\,Pr) \\ b_0^2 &= 4C_2/3a_0 \\ &= 20\nu\sqrt{[(20 + 21\,Pr)/35g\tau]}/Pr. \end{aligned}\right\} \quad (8)$$

This solution is

$$\left.\begin{aligned} u &= a_0(x + a)^{1/2} \\ \delta &= b_0(x + a)^{1/4} \end{aligned}\right\} \quad (9)$$

in which the parameter a depends upon the value that has been determined for either of the initial boundary conditions u_0 or δ_0. It is clear from the case discussed that zero would not be suitable selection for either u_0 or δ_0 and hence a is definitely different from zero. It is also clear that the solution is nowhere independent of the selected initial boundary condition. It may be pointed out here that if the transformation $\eta = C_1 y/(x + a)^{1/4}$; $\psi = 4\nu C_2(x + a)^{3/4}\zeta(\eta)$ had been used in the Pohlhausen solution of the problem, the additional initial boundary condition would also have been available and would have also effected a vertical shift in the boundary layer. This vertical shift or displacement has been observed experimentally and reported in [4]. The use of $a = 0$ in the latter treatment of the problem was evidently inappropriate.

If both initial boundary conditions are to be selected as independent quantities, then the solution takes a form

$$\left.\begin{aligned} u(x)/u(0) &= \bar{u}(X)/\bar{u}(X_0); \quad \bar{u}(X_0) \equiv \bar{u}_0 \\ \delta(x)/\delta(0) &= \bar{\delta}(X)/\bar{\delta}(X_0); \quad \bar{\delta}(X_0) \equiv \bar{\delta}_0 \\ X &= (x + a)/b \\ b &= [u(0)/\bar{u}(X_0)]^2/a_0^2 \\ &= [\delta(0)/\bar{\delta}(X_0)]^4/b_0^4 \\ a &= X_0[u(0)a_0\bar{u}(X_0)]^2 \\ &= X_0[\delta(0)/b_0\bar{\delta}(X_0)]^4 \\ a_0^2 &= 560g\tau/(20 + 21\,Pr) \\ b_0^2 &= 20\nu\sqrt{[(20 + 21\,Pr)/35g\tau]}/Pr \\ \bar{u}(X_0)/\bar{\delta}^2(X_0) & \\ &= [u(0)/\delta^2(0)]\nu(20 \\ &\qquad + 21\,Pr)/7g\tau Pr \end{aligned}\right\} \quad (10)$$

in which $\bar{u}(X)$, $\bar{\delta}(X)$ are solutions of the universal system

$$\left.\begin{aligned} \bar{u}'\bar{\delta}^2 + \bar{u}\bar{\delta}\bar{\delta}' &= \tfrac{3}{4} \\ 2\bar{u}\bar{u}'\,\bar{\delta}^2 + \bar{u}^2\,\bar{\delta}\bar{\delta}' & \\ - \frac{20 + 12\,Pr}{16}\,\bar{\delta}^2 + \frac{21\,Pr}{16}\,\bar{u} &= 0 \end{aligned}\right\} \quad (10a)$$

obtained from equations (5) and (6) by the transformation

$$\left.\begin{aligned} X &= (x + a)/b \\ \bar{u}(X) &= [u(x)/a_0]/b^{1/2} \\ \bar{\delta}(X) &= [\delta(x)/b_0]/b^{1/4}. \end{aligned}\right\} \quad (10b)$$

It may be shown that the system (10a) has the required universal solution

$$\begin{aligned} \bar{u}(X) &= X^{1/2}[1 \pm (2m - 3)/(3 - 4m)X^m \\ &\qquad + A_2/X^{2m} + \ldots \\ \bar{\delta}^2(X) &= X^{1/2}[1 \pm 1/X^m + B_2/X^{2m} + \ldots \\ m &= (60 + 63\,Pr)/16, \quad \bar{u}_0/\bar{\delta}_0^2 \neq 1 \\ &\bar{u}_0/\bar{\delta}_0^2 > 1, \quad \text{use } - \\ &\bar{u}_0/\bar{\delta}_0^2 < 1, \quad \text{use } + \end{aligned}$$

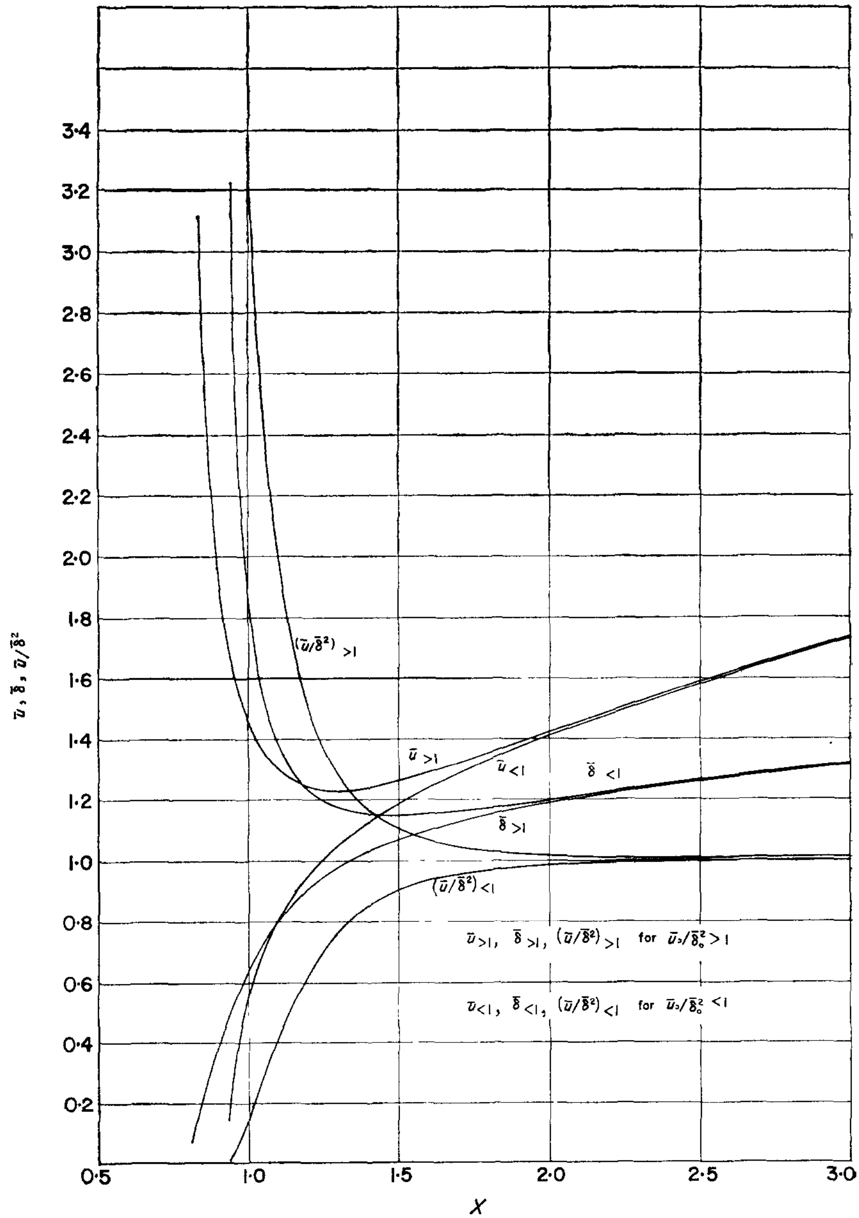

FIG. 1(d). Universal functions ($Pr = 0·7$) for arbitrary initial boundary conditions.

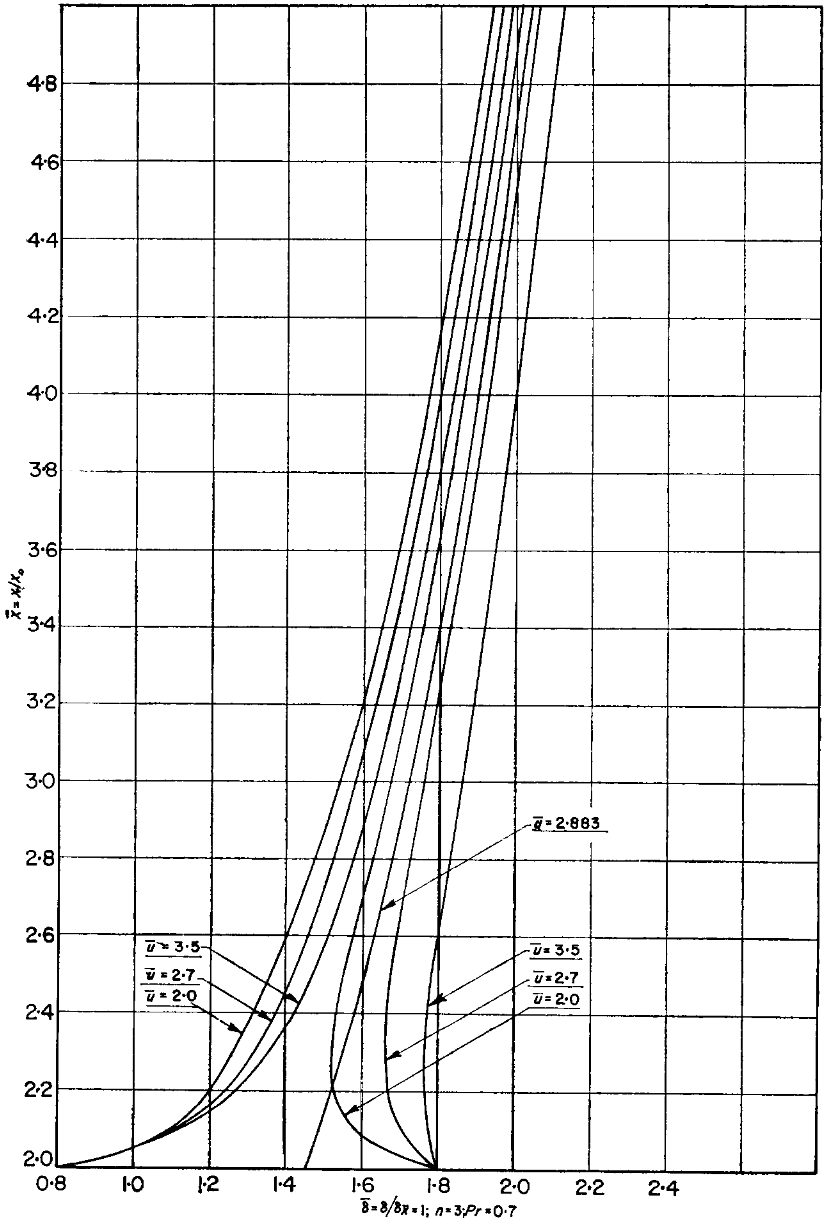

FIG. 2. Influence of initial boundary conditions on boundary layer thickness.

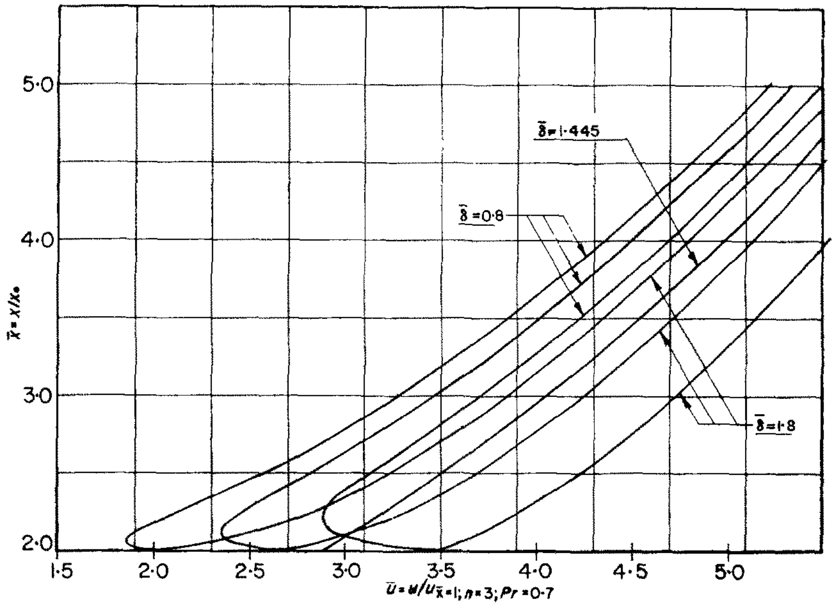

FIG. 3. Influence of initial boundary conditions on speed function.

in which the series coefficients are functions of *Pr*.

These results indicate that for arbitrary initial boundary conditions $u(0)$, $\delta(0)$ the solutions again acquire the values $\delta(x) = b_0(x + a)^{1/4}$, $u(x) = a_0(x + a)^{1/2}$ at distances adequately distant from the bottom $x = 0$ but the displacement parameter a is a function of the initial conditions assumed. Fig. 1(d) is a plot of the universal group $\bar{u}$, $\bar{\delta}$ and $\bar{u}/\bar{\delta}^2$ for $Pr = 0{\cdot}7$. When the values $u(0)$, $\delta(0)$ have been assigned then the last of the equations (10) determine $\bar{u}(X_0)/\bar{\delta}^2(X_0)$. The latter value determines an X_0 in the universal curve $\bar{u}/\bar{\delta}^2$. A complete solution of the specific problem is then readily computed by equations (10) and the universal curves $\bar{u}$, $\bar{\delta}$. Figs. 2 and 3 show some results obtained by direct numerical integration of equations (5) and (6) for several selections of the initial conditions. The values of u, δ were non-dimensionalized by standards computed in the third case solution.

Solution for the third case

As already indicated, this case includes a variable wall temperature setting. For the present purpose, a special variable wall temperature distribution function τ will be assumed (Fig. 4). We have

$$\left.\begin{aligned} \tau &= 0 & \bar{x} &\leqslant 0 \\ &= px^n = \bar{p}(x/x_0)^n = \bar{p}\bar{x}^n & 0 &\leqslant \bar{x} \leqslant 1 \\ &= \bar{p}[2 - (2 - \bar{x})^n] & 1 &\leqslant \bar{x} \leqslant 2 \\ &= 2\bar{p} & 2 &\leqslant \bar{x} \\ n &> 1 & & \end{aligned}\right\} \quad (12)$$

and the required leading edge condition, $(\mathrm{d}T_w/\mathrm{d}x)_{x=0} = 0$, is satisfied and a solution giving $U_{x=0} = 0$ is required.

One readily finds that equations (5) and (6) with the assigned temperature function τ do have the required solution

$$\left.\begin{aligned} u &= 2p^{1/2} \sqrt{\left[\frac{140}{4(3n+5) + 7\,Pr\,(5n+3)}\right]}\, x^{(1+n)/2} \\ \delta &= p^{-1/4}\left[\frac{60\nu}{(5n+3)\,Pr}\right]^{1/2} \times \left\{\frac{4(3n+5) + 7\,Pr\,(5n+3)}{35g}\right\}^{1/4} x^{(1-n)/4} \\ & \qquad 0 \leqslant x \leqslant x_0, \qquad n > 1 \end{aligned}\right\} \quad (13)$$

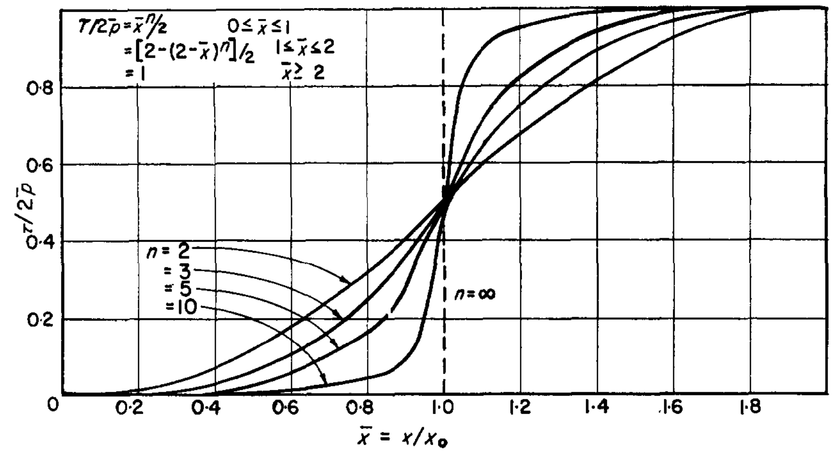

FIG. 4. Variations in wall temperature considered in surface bottom neighborhood.

Table 1. *Comparison of current results with those found by Pohlhausen's method*

$(Gr_x/4)^{1/4}y/x$	0	0·25	0·5	0·83	1·00	1·50	2·00	2·50	$Pr = 0·7$ $= 3$
$U/[4g\beta(T_w - T_\infty)x]^{1/2}$	0	0·08	0·128	0·148	0·144	0·094	0·030	0	
	0	0·08	0·125	0·140	0·135	0·105	0·07	0·04	Approx. values [3]
$(T - T_\infty)/(T_w - T_\infty)$	1	0·81	0·640	0·443	0·356	0·156	0·037	0	
	1	0·77	0·57	0·37	0·30	0·16	0·08	0·04	Approx. values [3]

The solution (13) was compared with that found in [5] as shown in Table 1. For the available comparable cases, that is for $n > 1$, it is clear that the results are not significantly different. For $n \leqslant 1$ new initial boundary conditions u_0, δ_0 are required and those cases are yet to be treated.

For values of $x > x_0$ the equations were integrated numerically for several values of $n > 1$. Some results are shown in Figs. 5 and 6. The values of u and δ for $n = 3$ $Pr = 0·7$ and $\bar{x} = 1$ were used as non-dimensionalizing standards. The Pohlhausen results with the bottom point at $\bar{x} = 1$ is included in Fig. 5 for comparison purposes. The parallelism of the δ curves soon after the constant temperature condition is reached is evident and would be expected in the light of the first cases studied. Fig. 7 is a detailed calculation for the case $n = 3$, $Pr = 0·7$ and a constant wall temperature section at $T_w = 0·1$ T_∞. It shows isotherms entering the wall bottom in an essentially horizontal attitude and thus pointing to the need in this region for the retention of the vertical heat conduction term in the energy equation (2). This effect is currently under investigation but some preliminary results may be given here.

BOUNDARY-LAYER EQUATIONS CONTAINING THE VERTICAL HEAT CONDUCTION TERM

With the inclusion of the vertical heat conduction term and the continued assumption of a similarity solution of the type (3) and (4), the

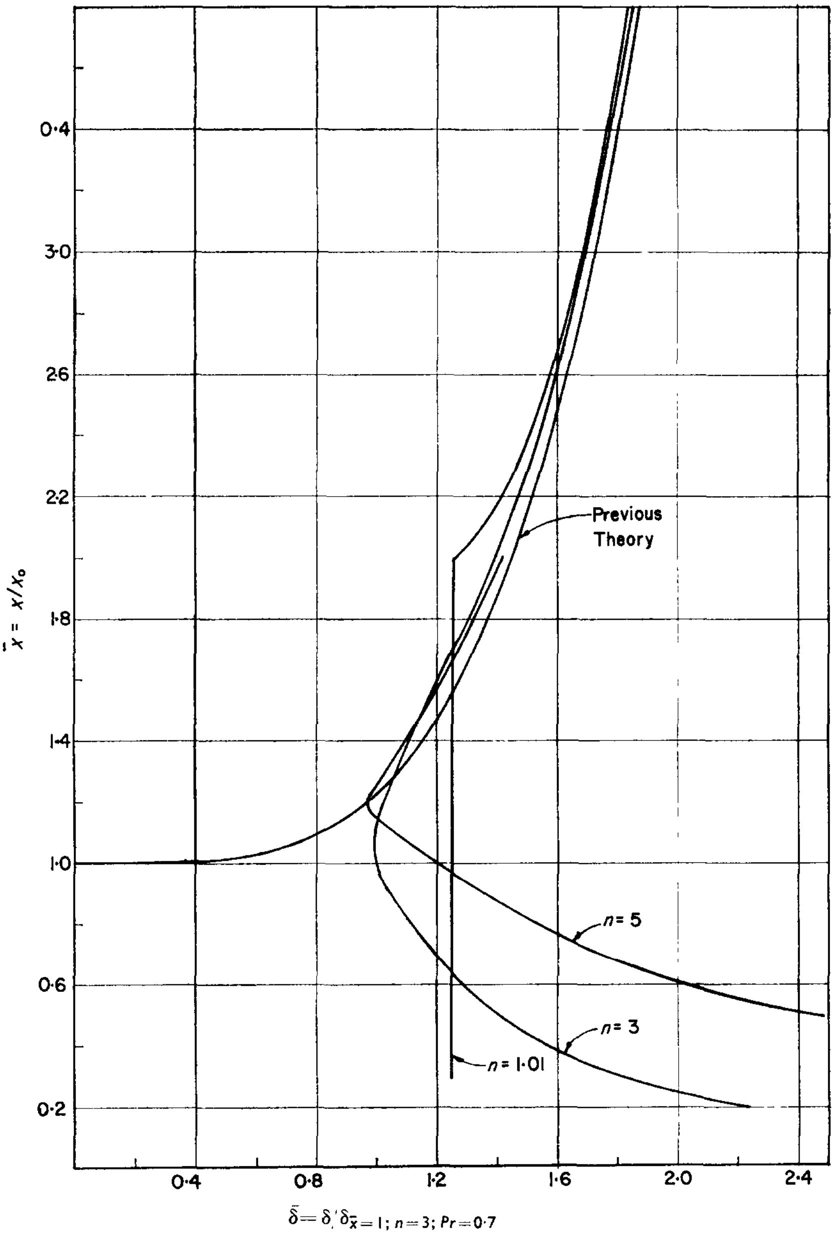

FIG. 5. Boundary layer thickness distributions for several values of n and previous theory.

boundary-layer differential equations take the form

$$\left.\begin{aligned} \tau' u\delta^2 + \tau(u'\delta^2 + u\delta\delta') - C_2(\tau + \delta^2\tau''/6 \\ + \tau'\delta\delta'/3 + \tau\delta\delta''/6) = 0 \\ 2uu'\delta^2 + u^2\delta\delta' - C_3\tau\delta^2 + C_4 u = 0 \end{aligned}\right\} \quad (14)$$

in which the coefficients C_2, C_3 and C_4 retain the values given in (7). For a wall temperature distribution function of the type $\tau = px^n$, $n > 1$ these equations have a solution of the form

$$\left.\begin{aligned} u = a_0 x^{(1+n)/2}\,[1 + a_1/x^{(n+3)/2} + a_2/x^{(n+3)} \\ + a_3/x^{3(n+3)/2} + \ldots \\ \delta = b_0 x^{(1-n)/4}\,[1 + b_1/x^{(n+3)/2} \\ + b_2/x^{(n+3)} + \ldots \end{aligned}\right\} \quad (15)$$

which indicates that at a suitable distance from the thermal leading edge which appears to require the vertical conduction term, the solution will be unaffected by its presence in the natural

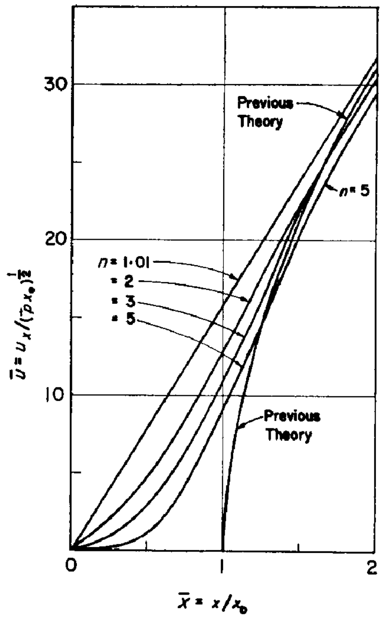

FIG. 6. Behavior of speed function factor in surface bottom neighborhood.

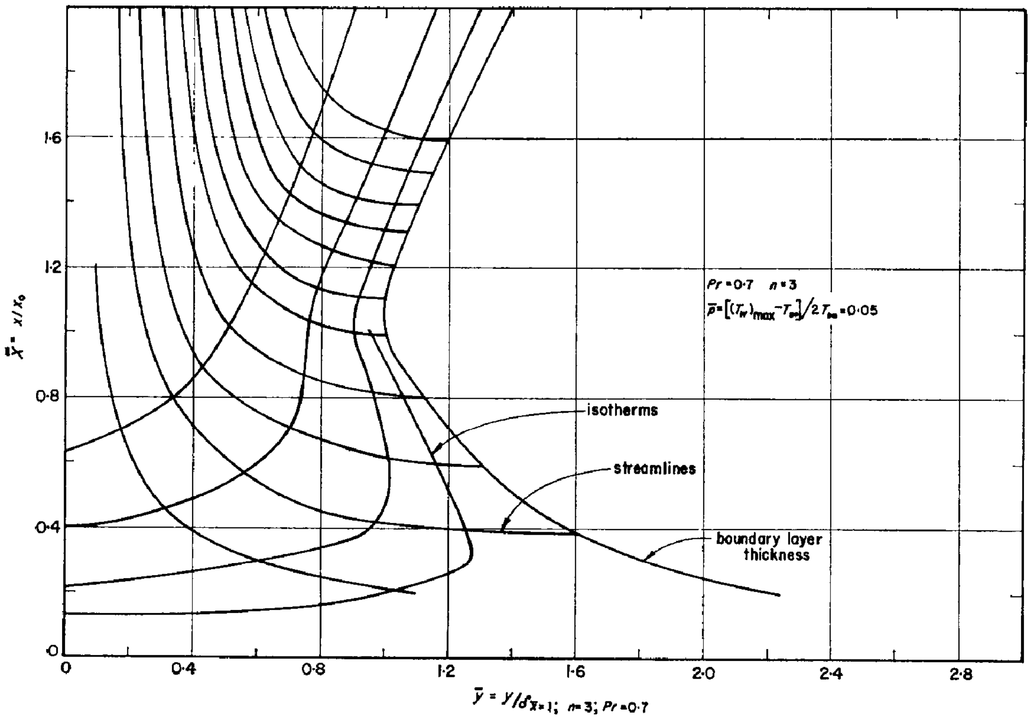

FIG. 7. Flow structure at surface bottom neighborhood. Mass flow between streamlines $[\text{Mass}/\rho_\infty(\delta u_x)_{\bar{x}=1;\,n=3;\,Pr=0\cdot7}] = 0{\cdot}02$.

convection equations. Equation (15) will be studied in detail for information about the correction effect of the vertical conduction term in the bottom region.

General variable wall temperature case

In cases where the similarity forms defined by equations (3) and (4) are admissible, a general variable wall temperature case may be treated by the solution setting equations (10), (10a) and (10b) in the following manner. Divide the wall into successive segments sufficiently small to allow the use of a mean constant temperature in each. The terminal conditions on each of the segments form the initial conditions on the succeeding segments. Once the initial conditions have been determined for the very bottom segment, the problem is completely solved by successive application of equations (10) and (10b). Several cases are now being studied in this manner.

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Résumé—La convection naturelle est étudiée au voisinage de plusieurs types de bords d'attaque thermiques de parois verticales. L'influence des conditions de bord d'attaque est déterminée pour des températures de paroi variables et uniformes particulières. Dans le cas d'une température de paroi constante, les conditions de bord d'attaque n'affectent, à des distances convenables au-dessus du bord d'attaque, que la position relative verticale de la couche limite et non son profil thermique ou dynamique. Des courbes universelles, qui peuvent être utilisées pour des conditions limites initales relativement arbitraires, sont présentées dans le cas où la température de paroi est constante et le nombre de Prandtl égal à 0,7. On montre comment utiliser ces courbes dans le cas de temperaturés de paroi relativement arbitraires.

Zusammenfassung—Die natürliche Konvektion an senkrechten Wänden in der Umgebung verschiedener Arten thermischer Anströmkanten wurde untersucht, sowohl für gleichmässige als auch für veränderliche Wandtemperaturen. Bei konstanter Wandtemperatur ergibt sich in angemessenem Abstand von der Anströmkante ein Einfluss nur auf die relative Vertikallage der Granzschicht, nicht auf die Form ihres Temperatur- oder Geschwindigkeitsprofils. Universelle Kurven, die für ziemlich willkürliche Anfangsbedingungen verwendbar sind, werden für gleichmässige Wandtemperatur und für die Prandtl-Zahl 0,7 angegeben. Zur Untersuchung verhältnismässig willkürlicher Wandtemperaturverteilungen wird gezeigt wie diese Kurven stückweise anzuwenden sind.

Аннотация—Рассматривается свободная конвекция от различных по форме обогреваемых кромок вертикальных стенок. Показано, какое влияние оказывают условия на кромке в случаях постоянной и изменяемой температуры стенки. При постоянной температуре стенки эти условия влияют на определенном расстоянии над кромкой только на относительное вертикальное положение пограничного слоя, а не на его профиль температуры или скорости. Для случая постоянной температуры стенки и числа Прандтля, равного 0,7, представлены универсальные кривые, которые могут быть использованы при относительно произвольных граничных условиях. Показана возможность использования этих кривых по частям при относительно произвольной температуре стенки.